

On Noether's theorem for the Euler-Poincaré equation on the diffeomorphism group with advected quantities

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In honor of Peter Olver's 60-th birthday

Abstract

We show how Noether conservation laws can be obtained from the particle relabelling symmetries in the Euler-Poincaré theory of ideal fluids with advected quantities. All calculations can be performed without Lagrangian variables, by using the Eulerian vector fields that generate the symmetries, and we identify the time-evolution equation that these vector fields satisfy. When advected quantities (such as advected scalars or densities) are present, there is an additional constraint that the vector fields must leave the advected quantities invariant. We show that if this constraint is satisfied initially then it will be satisfied for all times. We then show how to solve these constraint equations in various examples to obtain evolution equations from the conservation laws. We also discuss some fluid conservation laws in the Euler-Poincaré theory that do not arise from Noether symmetries, and explain the relationship between the conservation laws obtained here, and the Kelvin-Noether theorem given in Section 4 of Holm, Marsden and Ratiu, *Adv. in Math.*, 1998.

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1 Introduction

As Noether did in her famous paper [21], we are dealing with invariant variational principles. This subject has a vast literature and has been a favorite topic for Peter Olver, to which he returned many times [5, 22, 23, 24, 25, 26, 20].

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A Lie group transformation that leaves the Lagrangian invariant in Hamilton's principle is called a *variational Lie symmetry*. The correspondence between variational Lie symmetries and conservation laws for Euler-Lagrange equations is completely determined by the Noether's First Theorem [19, 21, 26]. Namely, every variational Lie symmetry yields a conservation law.¹ Our main goal here is to identify explicitly in terms of *Eulerian observables* the vector fields of the relabelling symmetry transformations under the Lie group G of smooth invertible maps that are responsible for some of the well-known conservation laws in the Euler-Poincaré theory of fluids with advected quantities [15]. In particular, we treat a few hands-on examples in fluid dynamics that recover some famous formulas such as helicity of fluids, Ertel's potential vorticity in geophysical fluid dynamics (GFD) and Chandrasekhar's cross-helicity for magnetohydrodynamics (MHD). We also discuss the relation of the classical Noether's Theorem with the Kelvin-Noether circulation theorem from the Euler-Poincaré theory of ideal fluids with advected quantities in [15]. In addition, we discuss conservation laws in the Euler-Poincaré theory that do not arise from Noether symmetries. Finally, we discuss some applications of Noether's Theorem in image registration problems.

It seems that every theoretical physicist and many mathematicians eventually feel compelled to write a paper about Noether's Theorem. Previous influential papers along similar lines about Noether's Theorem in fluid dynamics related to the directions taken here include [31, 9, 30, 12, 1, 27, 28, 6, 18] and of course references therein.

The main content of the paper is:

1. Section 2 briefly summarises the Euler-Poincaré formulation of ideal fluid dynamics with advected quantities. In particular, we summarise several simple but useful theorems that are available for studying how the Noether theorem associates variational Lie symmetries with conservation laws for fluids.
2. Section 3 uses these theorems in a sequence of examples that derive several of the most well-known conservation laws for ideal fluids in the Euler-Poincaré formulation [15].
3. Section 4 points out that not all fluid conservation laws follow from Noether's theorem, by considering the counterexample of magnetic helicity for MHD. It also makes a connection between Noether's theorem as discussed in this paper, and the Kelvin-Noether circulation theorem discussed in [15].
4. Section 5 discusses some numerical issues and applications of these ideas outside of fluid dynamics. Section 5 also raises topics for future research inspired by Lie symmetries and Noether's theorem.

2 Formulation

We begin by laying out the assumptions that underlie the Euler-Poincaré formulation. These are the following.

1. There is a right representation of the action of a Lie group G on its tangent space TG and on the vector space \mathcal{V} . The action on $TG \times \mathcal{V}$ is denoted by concatenation on the right, as $(v_g, a)h = (v_g h, ah)$ for $g, h \in G$.
2. The Lagrangian $L : TG \times \mathcal{V} \rightarrow \mathbb{R}$ is right G -invariant.
3. In particular, if $a_0 \in \mathcal{V}$, define the Lagrangian $L_{a_0} : TG \rightarrow \mathbb{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$. Then L_{a_0} is right-invariant under the lift to TG of the right action of G_{a_0} on G , where G_{a_0} is the isotropy group of a_0 .
4. Right G -invariance of the Lagrangian L permits us to define a *reduced Lagrangian* $l : \mathfrak{g} \times \mathcal{V} \rightarrow \mathbb{R}$ by

$$l(v_g g^{-1}, a_0 g^{-1}) = L(v_g, a_0). \quad (1)$$

Conversely, this relation defines for any $l : \mathfrak{g} \times \mathcal{V} \rightarrow \mathbb{R}$ a right G -invariant function $L : TG \times \mathcal{V} \rightarrow \mathbb{R}$.

5. For a curve $g(t) \in G$, let $u(t) := \dot{g}(t)g(t)^{-1}$ and define the curve $a(t) \in \mathcal{V}$ obtained from the action $G \times \mathcal{V} \rightarrow \mathcal{V}$ as the unique solution of the linear differential equation with time dependent coefficients

$$(\partial_t + \mathcal{L}_{u(t)}) a(t) = 0, \quad (2)$$

with initial condition $a(0) = a_0$ and Lie derivative $\mathcal{L}_{u(t)}$.

For fluids, the Lie group $G = \text{Diff}(\mathbb{R}^3)$ is the group of diffeomorphisms of three-dimensional space. This is the Lie group of smooth invertible maps defined on \mathbb{R}^3 and with smooth inverses.² At time t , the curve $g(t)$ defines

¹Noether's celebrated paper [21] contains two major theorems. The present paper discusses only the first of these theorems. For good discussions of the second Noether theorem, see e.g. [21, 26, 4, 19, 18].

²Strictly speaking, $G = \text{Diff}(\mathbb{R}^3)$ denotes the connected component at the identity of the diffeomorphisms. Its Lie algebra comprises the right-invariant vector fields on \mathbb{R}^3 , denoted as $\mathfrak{X}(\mathbb{R}^3)$.

the mapping from a reference configuration (known as *label space*) to the physical domain so that $x(t) = g(t)x_0$ describes Lagrangian particle trajectories for each label x_0 .

Definition 1. The solution $a(t) = a_0 g(t)^{-1}$ of equation (2) is called an *advected quantity* for fluids, and the right-invariant vector field $u(t) := \dot{g}g(t)^{-1} \in \mathfrak{X}(\mathbb{R}^3)$ is called the *Eulerian, or spatial, fluid velocity*.

Remark 2. Examples of advected quantities include the extensive thermodynamic properties that are carried by fluid elements such as their heat and mass. Equation (2) means physically that along the flow $g(t)$ of the vector field $u(t)$ the fluid elements are to be regarded as closed thermodynamic systems that do not exchange heat and mass with their neighbours.

Some particular examples of advected quantities that we discuss in this paper are:

1. Scalar fields (0-forms) $a(t) = s$ that satisfy:

$$(\partial_t + \mathcal{L}_{u(t)}) s = (\partial_t + \mathbf{u} \cdot \nabla) s = 0.$$

In geophysical models scalar advected quantities includes buoyancy due to heat and salinity.

2. Density fields (volume forms) $a(t) = \rho dV$ that satisfy

$$(\partial_t + \mathcal{L}_{u(t)}) \rho dV = (\partial_t \rho + \nabla \cdot (\mathbf{u} \rho)) dV = 0.$$

This type of advected quantity is used for the fluid density, or layer depth in shallow water models.

3. Flux fields (2-forms) $a(t) = \mathbf{B} \cdot d\mathbf{S}$ that satisfy

$$(\partial_t + \mathcal{L}_{u(t)}) \mathbf{B} \cdot d\mathbf{S} = (\partial_t \mathbf{B} - \text{curl}(\mathbf{u} \times \mathbf{B})) \cdot d\mathbf{S} = 0.$$

This type of advected quantity is used, e.g., for the magnetic flux in magnetohydrodynamics.

For more discussion of advected quantities, see [15]. The back-to-labels map that specifies the label of the fluid parcel currently at a given spatial position would also be an advected quantity. However, in this paper, we shall restrict ourselves to dealing only with *Eulerian observables*, and the particle label is not observable at any given Eulerian point in a fluid flow.

2.1 Euler-Poincaré theorem with advected quantities

Here we review the approach presented in [15] to obtaining the variational equation of motion, known as the Euler-Poincaré equation, for general reduced Lagrangians $l(u, a)$ with advected quantities.

Hamilton's principle $\delta S = 0$ for $S = \int l(u, a) dt$ with the reduced Lagrangian defined in equation (1) may be expressed either abstractly as

$$0 = \delta S = \delta \int_{t_0}^{t_1} l(u, a) dt = \int_{t_0}^{t_1} \left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta l}{\delta a}, \delta a \right\rangle dt, \quad (3)$$

where angle brackets denote appropriate pairings, or equivalently in coordinates with Lagrangian $\int_{\mathcal{D}} \ell(u, a) dV$

$$0 = \delta S = \delta \int_{t_0}^{t_1} \int_{\mathcal{D}} \ell(u, a) dV dt = \int_{t_0}^{t_1} \int_{\mathcal{D}} \left(\frac{\delta \ell}{\delta u} \cdot \delta u + \frac{\delta \ell}{\delta a} \delta a \right) dV dt, \quad (4)$$

where \mathcal{D} is the spatial domain with boundary $\partial\mathcal{D}$ on which the fluid velocity has no normal component; that is, $u \cdot n = 0$. The expressions $\frac{\delta l}{\delta u} \in \mathfrak{X}^*$ and $\frac{\delta l}{\delta a} \in \mathcal{V}^*$ are variational derivatives in u and a , respectively. We ensure that variations in g honour the boundary conditions, by defining $\delta g = w \circ g$, in which w is a vector field whose components satisfy $w \cdot n = 0$ on $\partial\mathcal{D}$.

In the remainder of the paper, we will find it convenient to use a hybrid notation that passes freely between the abstract notation and the more explicit coordinate notation, as in equations (3) and (4). We believe this hybrid notation, whose meaning will always be clear from the context, will appeal to a wider readership than the abstract notation. Conversely, we will sometimes find that the calculations we need to perform are written more directly in the abstract notation using the language of differential forms.

The infinitesimal transformations for u and a are [15]

$$\delta u = \dot{w} - \text{ad}_u w := \dot{w} + [u, w], \quad \delta a = -\mathcal{L}_w a. \quad (5)$$

Here a denotes any quantity that is advected with the flow, *e.g.* scalar tracers s , densities $\rho \, dV$ *etc.* The linear operator on w , ad_u , is defined in terms of $[u, w]$, which is the commutator (Lie bracket) of the vector fields u and w in $\mathfrak{X}(\mathbb{R}^3)$. Furthermore, we seek the stationary point $\delta S = 0$ in Hamilton's principle above, subject to $\delta g = 0$ at the endpoints $t = t_0$ and $t = t_1$; hence, we also require $w = \delta g g^{-1}$ to vanish at the endpoints.

Substitution in (3) now yields

$$0 = - \int_{t_0}^{t_1} \int_{\mathcal{D}} \left(\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \text{ad}_u^* \frac{\delta l}{\delta u} - \frac{\delta l}{\delta a} \diamond a \right) \cdot w \, dV \, dt + \left[\int_{\mathcal{D}} \frac{\delta l}{\delta u} \cdot w \, dV \right]_{t_0}^{t_1}, \quad (6)$$

where ad_u^* is the dual operator to ad_u defined by

$$\int_{\mathcal{D}} v \cdot \text{ad}_u^* m \, dV = \int_{\mathcal{D}} m \cdot \text{ad}_u v \, dV,$$

for all vector fields v , and whose explicit formula in components is

$$\text{ad}_u^* m = \nabla \cdot (u \otimes m) + (\nabla u)^T m.$$

This formula also happens to match the components of the Lie derivative for one-form densities [15],

$$\mathcal{L}_u(m \cdot dx \otimes dV) = (\nabla \cdot (u \otimes m) + (\nabla u)^T m) \cdot dx \otimes dV,$$

with line element dx and volume element dV . Notation for the diamond operation (\diamond) has also been introduced in equation (6). The diamond operation is defined by

$$\int_{\mathcal{D}} \left(\frac{\delta l}{\delta a} \diamond a \right) \cdot w \, dV := \int_{\mathcal{D}} \frac{\delta l}{\delta a} \cdot (-\mathcal{L}_w a) \, dV. \quad (7)$$

Vanishing of the first term in (6) for variations that are otherwise arbitrary now produces the Euler-Poincaré (EP) equation,

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \nabla \cdot \left(u \otimes \frac{\delta l}{\delta u} \right) + (\nabla u)^T \frac{\delta l}{\delta u} - \frac{\delta l}{\delta a} \diamond a = 0. \quad (8)$$

The EP equation in (8) is completed as an evolutionary system by including the equation of motion (2) for the advected quantities, a .

Noether's Theorem for Euler-Poincaré with advected quantities We consider symmetries of the action $S = \int l(u, a) dt$ that are obtained by infinitesimal transformations of the form $\delta g = \eta \circ g$ for a vector field η . Consequently, we have the infinitesimal transformations

$$\delta u = \dot{\eta} + [u, \eta], \quad \delta a = -\mathcal{L}_\eta a. \quad (9)$$

If the vector field η generates symmetries of the Lagrangian, then Hamilton's principle $\delta S = 0$ implies that

$$0 = - \int_{t_0}^{t_1} \int_{\mathcal{D}} \underbrace{\left(\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \nabla \cdot \left(u \otimes \frac{\delta l}{\delta u} \right) + (\nabla u)^T \frac{\delta l}{\delta u} - \frac{\delta l}{\delta a} \diamond a \right)}_{=0} \cdot \eta \, dV \, dt + \left[\int_{\mathcal{D}} \frac{\delta l}{\delta u} \cdot \eta \, dV \right]_{t_0}^{t_1}. \quad (10)$$

Here, the term in the time integral vanishes for solutions of the Euler-Poincaré equations, and we are left with the endpoint terms for arbitrary t_0 and t_1 . This implies Noether's theorem. Namely, a conservation law is associated with each vector field η that generates a symmetry of the Lagrangian [21]. These considerations prove the following.

Theorem 3 (Noether theorem for EP). *Each symmetry vector field η of the EP Lagrangian (3) for infinitesimal transformations given by (9) corresponds to an integral of the EP motion equation (8) satisfying*

$$\frac{d}{dt} \int_{\mathcal{D}} \frac{\delta l}{\delta u} \cdot \eta \, dV = 0, \quad (11)$$

for an appropriate inner product.

2.2 Relabelling symmetries

Let us now consider how to derive the vector fields η for the symmetry transformations in Noether's theorem in the case of fluids in the *Eulerian* representation. These symmetry transformations are called relabelling symmetries. They arise from the assumed right invariance of the EP Lagrangian $l(u, a)$ under the group G of diffeomorphisms (the Lie group of smooth invertible maps with smooth inverses). The Eulerian velocity $u(t) := \dot{g}(t)^{-1} \in \mathfrak{X}(\mathbb{R}^3)$ is right-invariant under this action and therefore it does not change under relabelling transformations. This invariance implies the following evolution equation for the vector field η :

$$\delta u = \dot{\eta} + [u, \eta] = 0, \quad (12)$$

where the bracket $[\cdot, \cdot]$ denotes commutation of vector fields. If a set of advected quantities $\{a\}$ exists, then the vector fields η for the symmetry transformations must also satisfy the additional conditions that

$$\delta a = -\mathcal{L}_\eta a = 0, \quad (13)$$

for each advected quantity a .

When there are no advected quantities present (as in the case of EPDiff [14], for example) equation (11) simply recovers the equation for conservation of momentum, as one sees from the following direct computation:

$$\begin{aligned} 0 &= \frac{d}{dt} \left\langle \frac{\delta l}{\delta u}, \eta \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\delta l}{\delta u}, \eta \right\rangle + \left\langle \frac{\delta l}{\delta u}, \frac{d}{dt} \eta \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\delta l}{\delta u}, \eta \right\rangle + \left\langle \frac{\delta l}{\delta u}, -[u, \eta] \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\delta l}{\delta u}, \eta \right\rangle + \left\langle \frac{\delta l}{\delta u}, \text{ad}_u \eta \right\rangle \\ &= \left\langle \frac{d}{dt} \frac{\delta l}{\delta u} + \text{ad}_u^* \frac{\delta l}{\delta u}, \eta \right\rangle \\ &= \left\langle \left(\frac{\partial}{\partial t} + \mathcal{L}_{u(t)} \right) \frac{\delta l}{\delta u}, \eta \right\rangle. \end{aligned}$$

In this computation, the angle brackets $\langle \cdot, \cdot \rangle$ denote the L^2 pairing $\mathfrak{X}^* \times \mathfrak{X} \rightarrow \mathbb{R}$ between the vector fields and their L^2 duals, the 1-form densities.

2.3 Theorems for advected quantities

We now develop general results for the case where one or more advected quantities are present. This requires determining whether all the conditions in (9) can be satisfied simultaneously. We shall conclude that, if they are satisfied initially, then they are satisfied for all times t , due to the commutative properties of Lie derivatives. This will enable us to derive conservation laws in various cases in the rest of the paper.

Theorem 4 (Commutator). *For any pair of smooth time-dependent vector fields $u(t), \eta(t) \in \mathfrak{X}$ and for any $a(t) \in V$ the following commutation relation holds among Lie derivatives,*

$$[\partial_t + \mathcal{L}_{u(t)}, \mathcal{L}_{\eta(t)}] a(t) = \mathcal{L}_{(\dot{\eta} + [u, \eta])} a(t). \quad (14)$$

Proof. For any $a(t) \in V$, one computes by the product rule for Lie derivatives that

$$(\partial_t + \mathcal{L}_{u(t)}) \mathcal{L}_\eta a(t) = \mathcal{L}_{(\dot{\eta} + [u, \eta])} a(t) + \mathcal{L}_\eta (\partial_t + \mathcal{L}_{u(t)}) a(t).$$

Hence, the commutation relation in (14) holds, and because $a(t) \in V$ is arbitrary, this implies the Lie derivative commutation relation

$$[\partial_t + \mathcal{L}_{u(t)}, \mathcal{L}_\eta] = \mathcal{L}_{(\dot{\eta} + [u, \eta])}. \quad (15)$$

□

Under the assumption that the variational vector field η satisfies the time-evolution equation (12) required for a relabelling symmetry transformation, one finds the following commutator theorem.

Corollary 5 (Symmetry). *If a vector field η satisfies equation (12) for an infinitesimal relabelling symmetry, then the Lie derivative \mathcal{L}_η commutes with the evolution operator, $(\partial_t + \mathcal{L}_{u(t)})$,*

$$[\partial_t + \mathcal{L}_{u(t)}, \mathcal{L}_{\eta(t)}] a(t) = 0 \quad \text{for} \quad \dot{\eta} + [u, \eta] = 0. \quad (16)$$

Proof. This symmetry corollary follows by inserting equation (12) into the commutation relation in equation (14). \square

Theorem 6 (Ertel theorem). *If the quantity a is advected as in equation (2) and the vector field η satisfies equation (12) for an infinitesimal relabelling symmetry, then $\mathcal{L}_\eta a$ is also advected.*

Proof. By equations (2) and (12) one finds the advection relation for $\mathcal{L}_\eta a$,

$$(\partial_t + \mathcal{L}_{u(t)}) \mathcal{L}_\eta a(t) = \mathcal{L}_\eta (\partial_t + \mathcal{L}_{u(t)}) a(t) = 0, \quad (17)$$

as a result of the condition (2) satisfied by advected quantities. \square

Consequently, if $\mathcal{L}_\eta a = 0$ in equation (13) holds initially, then it continues to hold under the EP flow. That is, we have the following.

Corollary 7 (Persistence). *If the vector field η is a relabelling symmetry, then the symmetry condition for advected quantities $\mathcal{L}_\eta a(t) = 0$ persists, provided it holds initially.*

Proof. If the left side of equation (17) vanishes initially at $t = 0$, then it continues to vanish for all time $t > 0$. \square

Definition 8 (Locally conserved quantities). *A locally conserved quantity $c(t)$ follows from the equations of motion and satisfies a local conservation law,*

$$(\partial_t + \mathcal{L}_{u(t)}) c(t) = 0, \quad (18)$$

which has the same form as an advection law.

Remark 9 (Local conservation laws). One distinguishes between advected quantities and locally conserved quantities. Namely, equations for advected quantities are obtained from the action $G \times V \rightarrow V$ and are independent of the fluid velocity. In contrast, local conservation laws involve the fluid velocity because they arise from the equations of motion.

Corollary 10 (Iterated conserved quantities). *If the quantity $c(t)$ satisfies a local conservation law (18) as a result of the EP equations of motion for all time, then $\mathcal{L}_\eta c(t)$ is also locally conserved for any relabelling symmetry.*

Proof. This follows from replacing $a(t)$ by $c(t)$ in equation (17). \square

Remark 11. *Iterating this process further is possible, but once a conserved quantity can be expressed in terms of advected quantities, iteration does not lead to new information.*

In the following section we adopt the strategy of constructing the symmetry vector fields η in Noether's theorem using the advected quantities that they preserve. This is accomplished in the examples below for several representative fluid flows in three dimensions. Occasionally, when enough freedom remains in the infinitesimal Lie symmetry η , the local conservation law that emerges from Noether's theorem may be re-substituted into the weak form of Noether's theorem to compute an additional integral conservation law.

3 Examples

3.1 Advected density: Conservation of vorticity and helicity

For the specific case that the mass density $a = \rho \, dV$ is advected and other advected quantities are absent, e.g., in barotropic fluid dynamics, the symmetry condition (13) is

$$\mathcal{L}_\eta(\rho \, dV) = d(\eta \lrcorner \rho \, dV) = 0.$$

Therefore, by Poincaré's Lemma, one may write locally that

$$\eta \lrcorner \rho \, dV = d(\Psi \cdot d\mathbf{x}) = \text{curl } \Psi \cdot d\mathbf{S}, \quad (19)$$

for some vector function Ψ .

For non-trivial topology (on a spherical annulus, for example), we may choose a simply connected patch bounded by a simple closed curve $C(t)$ that is transported by the fluid velocity u . We then restrict η at each time to the Lie algebra of vector fields that leave $C(t)$ invariant. (These vector fields η are tangent to the curve $C(t)$.) This choice allows us to define Ψ on the patch enclosed by $C(t)$ for each relabelling symmetry η .

Equation (19) the vector field η for each relabelling symmetry may be expressed in terms of a vector function Ψ , as

$$\eta = \rho^{-1} \text{curl } \Psi \cdot \nabla.$$

All such vector fields satisfy the advection condition (13) for the density $\rho \, dV$, since

$$\mathcal{L}_\eta(\rho \, dV) = \text{div}(\rho \rho^{-1} \text{curl } \Psi) \, dV = \text{div}(\text{curl } \Psi) \, dV = 0.$$

We substitute this solution for η into Noether's theorem and use Corollary 7 (persistence of the symmetry relation) to find,

$$\begin{aligned} 0 &= \frac{d}{dt} \left\langle \frac{\delta l}{\delta \mathbf{u}}, \eta \right\rangle \\ &= \frac{d}{dt} \int_{\mathcal{D}} \frac{\delta l}{\delta \mathbf{u}} \cdot \eta \, dV \\ &= \frac{d}{dt} \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge \eta \lrcorner (\rho \, dV) \\ \text{By (19)} &= \frac{d}{dt} \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge d(\Psi \cdot d\mathbf{x}) \\ &= \int_{\mathcal{D}} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\Psi \cdot d\mathbf{x}) + \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge \frac{\partial}{\partial t} d(\Psi \cdot d\mathbf{x}) \\ \text{By (17)} &= \int_{\mathcal{D}} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\Psi \cdot d\mathbf{x}) + \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge (-d\mathcal{L}_u(\Psi \cdot d\mathbf{x})) \\ &= - \int_{\mathcal{D}} \left(\frac{\partial}{\partial t} d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) + \mathcal{L}_u d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \right) \wedge (\Psi \cdot d\mathbf{x}) \\ &= - \int_{\mathcal{D}} \left(\frac{\partial}{\partial t} + \mathcal{L}_{u(t)} \right) d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge (\Psi \cdot d\mathbf{x}). \end{aligned}$$

Since Ψ is arbitrary, vanishing of the final line gives the weak form of the **local conservation of the vorticity 2-form**,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) \left(\text{curl} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{S} \right) = 0, \quad (20)$$

where vorticity is defined as the curl of the specific momentum (momentum per unit mass). The specific momentum is equal to the velocity for Euler fluids; so its curl in that case is the usual Euler fluid vorticity.

Remark 12 (Two velocity vectors). Two velocity vectors appear in the computations above: These are the fluid velocity vector \mathbf{u} in the Lie derivative \mathcal{L}_u and the specific momentum covector $\rho^{-1} \delta l / \delta \mathbf{u}$ in the 1-form $\rho^{-1} \delta l / \delta \mathbf{u} \cdot d\mathbf{x}$. These two velocities are the basic ingredients for performing modelling and analysis in any ideal fluid problem. They appear together and have separate meanings in the Euler-Poincaré equation and throughout the present paper, as illustrated in the examples below.

Example 13 (Incompressible Euler equations). *The incompressible Euler equations have reduced Lagrangian*

$$l(u, \rho) = \int_{\mathcal{D}} \frac{\rho |u|^2}{2} + p(1 - \rho) \, dV,$$

where p is a Lagrange multiplier enforcing incompressibility $\rho = 1$. The variational derivatives are

$$\frac{\delta l}{\delta u} = \rho u, \quad \frac{\delta l}{\delta \rho} = \frac{|u|^2}{2} - p.$$

In this case, the conserved vorticity is

$$\operatorname{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta u} \right) = \operatorname{curl} u.$$

Example 14 (Incompressible Euler-alpha equations). *The incompressible Euler-alpha equations with $\rho = 1$ have reduced Lagrangian*

$$l(u, \rho) = \int_{\mathcal{D}} \frac{\rho}{2} (|u|^2 + \alpha^2 |\nabla u|^2) + p(1 - \rho) \, dV.$$

The variational derivatives are

$$\frac{\delta l}{\delta u} = \rho u - \alpha^2 \nabla \cdot \rho \nabla u, \quad \frac{\delta l}{\delta \rho} = \frac{1}{2} (|u|^2 + \alpha^2 |\nabla u|^2) - p.$$

In this case, the conserved vorticity is

$$\operatorname{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta u} \right) = \operatorname{curl} u - \frac{\alpha^2}{\rho} \nabla \cdot \rho \nabla u,$$

which becomes $\operatorname{curl}(u - \alpha^2 \nabla^2 u)$ since $\rho = 1$.

In the preserved 2-form $d(\Psi \cdot d\mathbf{x})$ introduced in equation (19), the vector function Ψ is determined (locally) for each choice of symmetry vector field η . Likewise, we have seen that each choice of the vector function Ψ corresponds to a certain relabelling symmetry η . Consequently, Corollary 7 for the persistence of symmetry and the definition of a local conservation law in equation (18) would allow us to replace $d(\Psi \cdot d\mathbf{x})$ with another conserved 2-form. In particular, we may choose the conserved vorticity 2-form and set,

$$d(\Psi \cdot d\mathbf{x}) = d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right).$$

After this identification, we may draw the conclusion from persistence in Corollary 7 that

$$\begin{aligned} 0 &= \frac{d}{dt} \left\langle \frac{\delta l}{\delta \mathbf{u}}, \eta \right\rangle \\ &= \frac{d}{dt} \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge \eta \lrcorner \rho \, dV \\ &= \frac{d}{dt} \int_{\mathcal{D}} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\Psi \cdot d\mathbf{x}) \\ &= \frac{d}{dt} \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \\ &= \frac{d}{dt} \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot \operatorname{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) dV. \end{aligned}$$

Thus, the weak form of the local conservation law for vorticity yields a conservation law for the **helicity** integral,

$$H := \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot \operatorname{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) dV. \quad (21)$$

The vector field for the symmetry associated with helicity conservation is,

$$\eta_H = \rho^{-1} \operatorname{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) \cdot \nabla. \quad (22)$$

The characteristic paths of the vector field η_H may be regarded as vortex lines, and these satisfy the symmetry condition (12), as a result of the EP equation (8). That is, the characteristic paths of η_H are frozen into the flow of the fluid velocity. This means that shifts along these paths are relabelling symmetries and the corresponding Noether conservation law is the helicity H in equation (21). In particular, the symmetry associated with conservation of helicity is a relabelling of the frozen-in vortex lines.

As mentioned earlier, the momentum per unit mass is equal to the velocity \mathbf{u} for Euler fluids, so that conservation of the helicity for Euler fluids may be expressed as

$$\frac{d}{dt} \int_{\mathcal{D}} \mathbf{u} \cdot \text{curl } \mathbf{u} \, dV = 0.$$

The spatial integral H defining the fluid helicity in (21) measures the knottedness, or number of linkages, of the vortex lines, that is, lines of $\boldsymbol{\omega} = \text{curl}(\rho^{-1} \delta l / \delta \mathbf{u})$. This fluid helicity indicates the topological complexity of the winding of the vortex lines in $\boldsymbol{\omega}$ amongst themselves in the spatial domain [3]. Physically, helicity conservation arises because the vortex lines are frozen into the flow of the diffeomorphisms and, thus, they cannot unknot.

Remark 15 (Ertel's theorem in hydrodynamic notation). We identify the evolution operator in Theorem 6 as the familiar Lagrangian time derivative D/Dt ,

$$\partial_t + \mathcal{L}_{u(t)} = \frac{D}{Dt},$$

and we express the vector field η_H in equation (22) in terms of a generalised vorticity vector $\boldsymbol{\omega}$, defined as

$$\eta_H = \rho^{-1} \text{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) \cdot \nabla =: \rho^{-1} \boldsymbol{\omega} \cdot \nabla \quad \text{with} \quad \boldsymbol{\omega} := \text{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right).$$

Introducing this familiar hydrodynamic notation allows one to write the symmetry relation (16) in the case for the action of the Lie derivative on a function $a(t)$ as

$$\frac{D}{Dt} (\rho^{-1} \boldsymbol{\omega} \cdot \nabla) a(t) = (\rho^{-1} \boldsymbol{\omega} \cdot \nabla) \frac{D}{Dt} a(t), \quad (23)$$

which is the usual form of the classical Ertel theorem [10]. For a scalar advected function, $a \in \Lambda^0$, Corollary 7 (persistence) yields yet another scalar conservation law, for $q = (\rho^{-1} \boldsymbol{\omega} \cdot \nabla) a$.

3.2 Advected density and tracer: Conservation of potential vorticity

Proposition 16. *For the case of two advected quantities $a_1 = \rho \, dV \in \Lambda^3$, $a_2 = s \in \Lambda^0$, the simultaneous solution of $\mathcal{L}_\eta \rho \, dV = 0$ and $\mathcal{L}_\eta s = 0$ is*

$$\eta \lrcorner \rho \, dV = d(\phi \, ds), \quad (24)$$

for general $\phi \in \Lambda^0$.

The proof of this proposition is simple, because a_1 is a top form and a_2 is a bottom form.

Proof. Symmetry requires that these two advected quantities satisfy

$$\eta \lrcorner \rho \, dV = d(\boldsymbol{\Psi} \cdot d\mathbf{x}) \quad \text{and} \quad \eta \lrcorner ds = \eta \cdot \nabla s = 0.$$

Thus,

$$ds \wedge \eta \lrcorner \rho \, dV = (\nabla s \cdot \eta) \rho \, dV = 0,$$

and, hence,

$$0 = (\nabla s \cdot \eta) \rho \, dV = ds \wedge (\eta \lrcorner \rho \, dV) = ds \wedge d(\boldsymbol{\Psi} \cdot d\mathbf{x}) = ds \wedge d(\phi \, ds).$$

□

The two advected quantities $d(\boldsymbol{\Psi} \cdot d\mathbf{x})$ and $d(\phi \, ds)$ both equal $(\eta \lrcorner \rho \, dV)$, so they satisfy the same evolution equation. In particular, the following advection equation holds

$$\frac{\partial}{\partial t} d(\phi \, ds) = -d\mathcal{L}_u(\phi \, ds).$$

Now one may substitute $\eta \lrcorner \rho \, dV = d(\phi \, ds)$ into the Noether theorem calculation as above and recompute, finding this time that:

$$\begin{aligned}
0 &= \frac{d}{dt} \left\langle \frac{\delta l}{\delta \mathbf{u}}, \eta \right\rangle \\
&= \frac{d}{dt} \int_{\mathcal{D}} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge (\eta \lrcorner \rho \, dV) \\
\text{By (24)} &= \frac{d}{dt} \int_{\mathcal{D}} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\phi \, ds) \\
\text{By (17)} &= \int_{\mathcal{D}} \frac{\partial}{\partial t} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\phi \, ds) + \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge (-d\mathcal{L}_u(\phi \, ds)) \\
&= - \left\langle \frac{\partial}{\partial t} d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge ds + \mathcal{L}_u d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge ds, \phi \right\rangle \\
\text{By } s = a_2 &= - \left\langle \frac{\partial}{\partial t} \left(d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge ds \right) + \mathcal{L}_u \left(d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge ds \right), \phi \right\rangle \\
&= - \left\langle \left(\frac{\partial}{\partial t} + \mathcal{L}_{u(t)} \right) \left(d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge ds \right), \phi \right\rangle.
\end{aligned}$$

As before, all boundary terms have been dropped in spatial integrations by parts. Since ϕ is arbitrary, the final line of the calculation above gives the weak form of the conservation law for Ertel **potential vorticity** (PV) density, defined as [10],

$$q \rho \, dV := d \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge ds = \text{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) \cdot \nabla s \, dV. \quad (25)$$

The corresponding local conservation law is

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) (q \rho \, dV) = 0. \quad (26)$$

The arbitrary function ϕ in the weak form of the local conservation law for potential vorticity also yields the integral conservation law,

$$\frac{d}{dt} \int_{\mathcal{D}} \Phi(q) \rho \, dV = 0, \quad (27)$$

in which Φ is an arbitrary function and we used $\partial_t(\Phi(q)\rho) = -\nabla \cdot (\Phi(q)\rho)$.

The vector field for the symmetry associated with PV conservation in (26) may be computed from equation (24) as

$$\eta_{PV} = \rho^{-1} (\nabla \phi \times \nabla s) \cdot \nabla, \quad (28)$$

and it represents shifts along level sets of s . In particular, the characteristic paths of the vector field η_{PV} lie along the level sets of s , which in turn are frozen into the flow of the fluid velocity. This means that shifts along the characteristic paths of η_{PV} are relabelling symmetries and the corresponding Noether conservation law is the advection of the potential vorticity q in equation (25).

Example 17 (Rotating Euler-Boussinesq equations). *The reduced Lagrangian for the rotating Euler-Boussinesq equations is*

$$l = \int_{\mathcal{D}} \rho \frac{|u|^2}{2} + \rho u \cdot R - zb + p(1 - \rho) \, dV,$$

where b is the buoyancy satisfying $\frac{\partial b}{\partial t} + \mathcal{L}_u b = 0$, and where R satisfies $\text{curl } R = 2\mathbf{\Omega}$. Consequently, the conserved potential vorticity may be computed, as follows:

$$\text{curl} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) = \text{curl}(\mathbf{u} + \mathbf{R}) \quad \text{with} \quad \text{curl } \mathbf{R} = 2\mathbf{\Omega} \quad \implies \quad q = (\text{curl } \mathbf{u} + 2\mathbf{\Omega}) \cdot \nabla b.$$

3.3 Advected density and flux (2-form): Conservation of cross helicity

Proposition 18. *For the case that $a_1 = \rho \, dV \in \Lambda^3$ and $a_2 = \mathbf{B} \cdot d\mathbf{S} = d(\mathbf{A} \cdot d\mathbf{x}) \in \Lambda^2$, the only simultaneous solution of $\mathcal{L}_\eta \rho \, dV = 0$ and $\mathcal{L}_\eta \mathbf{B} \cdot d\mathbf{S} = 0$ is*

$$\eta \lrcorner \rho \, dV = \mathbf{B} \cdot d\mathbf{S}. \quad (29)$$

Proof.

$$\text{Recall } \eta \lrcorner \rho \, dV = d(\Psi \cdot d\mathbf{x}) \text{ and identify } d(\Psi \cdot d\mathbf{x}) = \mathbf{B} \cdot d\mathbf{S}.$$

□

In this case, Noether's theorem implies the conserved quantity

$$\begin{aligned} 0 &= \frac{d}{dt} \left\langle \frac{\delta l}{\delta \mathbf{u}}, \eta \right\rangle \\ &= \frac{d}{dt} \int_{\mathcal{D}} \left(\frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge (\eta \lrcorner \rho \, dV) \\ \text{By (29)} &= \frac{d}{dt} \int_{\mathcal{D}} \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge \mathbf{B} \cdot d\mathbf{S} \\ &= \frac{d}{dt} \int_{\mathcal{D}} \left(\mathbf{B} \cdot \frac{1}{\rho} \frac{\delta l}{\delta \mathbf{u}} \right) dV. \end{aligned}$$

This is the **cross helicity**, which is known to be conserved, in particular, for ideal magnetohydrodynamics (MHD) [3]. The vector field for the symmetry associated with conservation of cross helicity is,

$$\eta_{CH} = \rho^{-1} \mathbf{B} \cdot \nabla,$$

which represents a field of shifts along magnetic field lines. The characteristic paths of the vector field η_{CH} are magnetic field lines that satisfy the symmetry condition (12), as a result of the advection equation (13) for magnetic flux $\mathbf{B} \cdot d\mathbf{S}$. That is, the characteristic paths of η_{CH} are frozen into the flow of the fluid velocity. This means that shifts along these characteristic paths are relabelling symmetries and the corresponding Noether conservation law is the cross helicity.

4 Other conservation laws for ideal fluids

In this section, we first point out that not all fluid conservation laws follow from Noether's theorem, as formulated above, by considering the counterexample of magnetic helicity for MHD. We then make a connection between the Noether's theorem discussed in this paper, and the Kelvin-Noether circulation theorem discussed in [15].

4.1 Magnetic helicity

The distinction between advected quantities and locally conserved quantities comes back into play, when one considers compound advected quantities that are conserved independently of the motion equation. For example advection of the scalar s and the exact 2-form $\mathbf{B} \cdot d\mathbf{S} = d(\mathbf{A} \cdot d\mathbf{x})$ lead immediately to advection of the compound quantities,

$$ds \wedge \mathbf{B} \cdot d\mathbf{S} = \text{div}(s\mathbf{B}) \, dV \quad \text{and} \quad \mathbf{A} \cdot d\mathbf{x} \wedge \mathbf{B} \cdot d\mathbf{S} = \mathbf{A} \cdot \mathbf{B} \, dV.$$

The former is trivial, because its integral over space vanishes identically. However, the latter is the famous **magnetic helicity**, whose spatial integral measures the knottedness, or number of linkages, of the magnetic field lines. That is, the magnetic helicity indicates the topological complexity of the winding of the magnetic field lines amongst themselves in the spatial domain. The preservation of this magnetic winding number is a fascinating property of ideal MHD flows [3], but it does not arise from a Noether symmetry. It arises here as a compound Lagrangian quantity whose Eulerian interpretation is deep and interesting. It is beyond our present scope for further study, except to provide a counterexample to the conjecture that a converse of Noether's theorem might exist for Euler-Poincaré ideal fluid theories.

4.2 Modified vorticity, potential vorticity, helicity and the Kelvin-Noether theorem

More general conservation laws can be obtained by expanding the set of variables, so that the time variation of the quantity a is enforced by a Lagrange multiplier b (known as a Clebsch variable) instead of constraining the variation δa . As described in [8, 7, 11], the same Euler-Poincaré equations are obtained this way. In this case, Hamilton's principle becomes

$$S_{Clebsch} = \int_{t_0}^{t_1} l(u, a) + \left\langle b, \left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) a \right\rangle dt, \quad (30)$$

with Lagrange multiplier b to be determined. Then, Hamilton's principle yields after a short calculation,

$$\begin{aligned} 0 = \delta S_{Clebsch} &= \int_{t_0}^{t_1} \left\langle \frac{\delta l}{\delta u} - b \diamond a, \delta u \right\rangle + \left\langle \delta b, \left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) a \right\rangle \\ &\quad + \left\langle \frac{\delta l}{\delta a} - \left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) b, \delta a \right\rangle dt + \left[\langle b, \delta a \rangle \right]_{t_0}^{t_1}. \end{aligned}$$

We now consider symmetries of the form

$$\delta u = \dot{\eta} + [u, \eta], \quad \delta a = 0, \quad \delta b = 0,$$

for a general relabelling vector field η that satisfies $\dot{\eta} + [u, \eta] = 0$, but is not constrained to be a symmetry of the quantity a . Noether's theorem then leads to

$$\frac{d}{dt} \left\langle \frac{1}{\rho} \frac{\delta l}{\delta u}, \eta \lrcorner \rho dV \right\rangle = 0.$$

Next, we define the $(\tilde{\diamond})$ operation in terms of the diamond operation by

$$b \diamond a =: \left(\frac{b}{\rho} \tilde{\diamond} a \right) \otimes \rho dV. \quad (31)$$

The $(\tilde{\diamond})$ operation allows one to express a 1-form density as the product of a 1-form and the advected mass density.

After a calculation similar to that leading to the result (20), one may write the vanishing of the η -coefficient in the previous variational equation for $\delta S_{Clebsch} = 0$ as

$$\frac{d}{dt} \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta u} - \frac{d}{dt} \oint_{\gamma_t} \frac{b}{\rho} \tilde{\diamond} a + \oint_{\gamma_t} \left(\frac{1}{\rho} \frac{\delta l}{\delta a} \right) \tilde{\diamond} a - \oint_{\gamma_t} \left(\frac{1}{\rho} (\partial_t + \mathcal{L}_u) b \right) \tilde{\diamond} a = 0, \quad (32)$$

which is found after substituting $(\frac{\partial}{\partial t} + \mathcal{L}_u) a = 0$, as imposed by the δb -variation. In the loop integrals, the closed circuit γ_t moves with the flow of the fluid velocity vector field u .

Now we have two choices. Namely, we may either eliminate Lagrange multiplier b by using the variational equation for b ,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) b = \frac{\delta l}{\delta a}, \quad (33)$$

or we may keep b as an additional dynamical variable satisfying equation (33). The first choice yields the Kelvin-Noether theorem of [15], and the second choice yields an advection equation for a quasi-vorticity vector field, plus the additional equation (33) for b . Specifically, in the first choice, the second and fourth terms in equation (32) cancel, leaving

$$\frac{d}{dt} \oint_{\gamma_t} \frac{1}{\rho} \frac{\delta l}{\delta u} + \oint_{\gamma_t} \left(\frac{1}{\rho} \frac{\delta l}{\delta a} \right) \tilde{\diamond} a = 0, \quad (34)$$

which is the Kelvin-Noether theorem [15] for circulation.

In the second choice, the third and fourth terms in equation (32) cancel instead, thereby leaving the following circulation conservation law,

$$\frac{d}{dt} \oint_{\gamma_t} \left(\frac{1}{\rho} \frac{\delta l}{\delta u} - \frac{b}{\rho} \tilde{\diamond} a \right) = 0, \quad (35)$$

or, equivalently in vector notation,

$$\frac{d}{dt} \oint_{\gamma_t} \tilde{\mathbf{u}} \cdot d\mathbf{x} = 0, \quad \text{with} \quad \tilde{\mathbf{u}} \cdot d\mathbf{x} := \left(\frac{1}{\rho} \frac{\delta l}{\delta u} - \frac{b}{\rho} \tilde{\diamond} a \right). \quad (36)$$

The price for this circulation conservation law is that the Lagrange multiplier b remains and satisfies equation (33), instead of being eliminated. However, keeping b as a dynamical variable also has the added value that doing so yields a quasi-vorticity $\tilde{\omega}$ involving b that satisfies the advection law for a 2-form under the flow of the velocity vector field, u ,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u\right)(\tilde{\omega} \cdot d\mathbf{S}) = 0, \quad \text{with} \quad \tilde{\omega} \cdot d\mathbf{S} = (\text{curl } \tilde{\mathbf{u}}) \cdot d\mathbf{S} := d\left(\frac{1}{\rho} \frac{\delta l}{\delta u} - \frac{b}{\rho} \tilde{\diamond} a\right). \quad (37)$$

Moreover, the vector field $\rho^{-1}\tilde{\omega}$, which is derived via the relation

$$\rho^{-1}\tilde{\omega} \lrcorner \rho dV = \tilde{\omega} \cdot d\mathbf{S},$$

also satisfies the invariance equation (12), namely,

$$\frac{\partial}{\partial t}(\rho^{-1}\tilde{\omega}) + [u, \rho^{-1}\tilde{\omega}] = 0, \quad \text{which means} \quad \left[\frac{\partial}{\partial t} + \mathcal{L}_u, \mathcal{L}_{(\rho^{-1}\tilde{\omega})}\right] = 0, \quad (38)$$

as demonstrated in equation (15) in the proof of Theorem 4.

Consequently, keeping the Lagrange multiplier b as a dynamical variable produces an Ertel theorem of the form (23) and yields conservation laws for the corresponding potential vorticity and helicity.

Moreover, these equations apply for essentially *any* fluid theory; so keeping the Lagrange multiplier b instead of eliminating it affords a certain universality to the formulation. The corresponding conserved potential quasi-vorticity \tilde{q} and quasi-helicity \tilde{H} are defined by

$$\tilde{q} := \rho^{-1}\tilde{\omega} \cdot \nabla s \quad \text{with} \quad \tilde{H} := \int_{\mathcal{D}} \tilde{\mathbf{u}} \cdot d\mathbf{x} \wedge d(\tilde{\mathbf{u}} \cdot d\mathbf{x}) = \int_{\mathcal{D}} \tilde{\omega} \cdot \text{curl}^{-1} \tilde{\omega} dV, \quad (39)$$

in terms of the quasi-vorticity $\tilde{\omega}$ in (37) and writing \tilde{q} for an advected scalar function, $a_1 = s$, and mass density, $a_2 = \rho dV$.

Example 19. As an example, consider the *particular* case $a_1 = s$ and $a_2 = \rho dV$, when an analogue of potential vorticity exists. In this case, the quasi-vorticity 2-form is given by

$$\tilde{\omega} \cdot d\mathbf{S} = d((\mathbf{u} + \rho^{-1}b \nabla s) \cdot d\mathbf{x}) = \text{curl}(\mathbf{u} + \rho^{-1}b \nabla s) \cdot d\mathbf{S},$$

and the Lagrange multiplier b satisfies

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \frac{b}{\rho} = \frac{1}{\rho} \frac{\delta l}{\delta s}.$$

Thus, at the cost of keeping the b -equation as $(\frac{\partial}{\partial t} + \mathcal{L}_u)b = \delta l / \delta a$, we can extend the Kelvin, Ertel and helicity theorems for most fluid theories, but these are Noether conservation laws for the Lagrangian corresponding to the extended variable set (u, a, b) .

5 Conclusions

In this paper we showed how to obtain conserved quantities for ideal fluid models that can be obtained from the Euler-Poincaré equations with advected quantities. The conserved quantities are obtained *via* Noether's Theorem from relabelling symmetries of the Lagrangian, which are generated by vector fields that satisfy the condition $\eta_t - \text{ad}_u \eta = 0$. Fluid theories usually involve advected quantities that evolve according to the equation $a_t + \mathcal{L}_u a = 0$: an advected density is almost always present, and other possibilities include advected scalars such as buoyancy, or advected 2-forms such as magnetic flux. In order to have a symmetry of the Lagrangian it is also necessary to satisfy equation (13) so that the vector field η generates a symmetry of the advected quantity $a(t)$. In Corollary 7 we showed that if (13) is satisfied initially then it is satisfied for all subsequent times. In general, defining a parameterisation of the null space of \mathcal{L}_η for η then leads to evolution equations defined on the dual of the space of parameterising functions. Note that this approach is different to that of Section 4 of [15], which does not use Noether's Theorem and instead performs computations on the Euler-Poincaré equations directly.

In this paper we considered fluid theories with advected density, leading to conservation of vorticity, as well as advected density plus advected tracers, leading to conservation of potential vorticity, and advected density plus

advected 1-forms (such as magnetic flux), leading to global conservation of cross-helicity. There are many other advected quantities that could be considered, notably advected tensor fields which are used in the theory of ideal complex fluids [13]. However, since most fluid theories include an advected density (even incompressible flow, for which the pressure is a Lagrange multiplier that enforces that the density remain constant), it is not always possible to simultaneously solve all of the constraints on η arising from the requirement that η generates symmetries of all of the advected quantities involved. For example, when a density and a magnetic flux are both present, there is only one symmetry and hence only one globally conserved quantity. An interesting class of problems in which density is not necessarily present arise in computational anatomy [17]. Here the aim is to find the solution of the EPDiff equation (for which the reduced Lagrangian is a functional of u only) which transports one configuration of an advected quantity a to another. The advected quantity might be a scalar (for greyscale images), a singular measure (for curves and surfaces), or even a tensor field (for diffusion tensor images). Solutions of the EPDiff equation are geodesics on the diffeomorphism group; for these solutions to drop to geodesics on the shape space corresponding to the chosen advected quantities, all of the conserved Noether quantities must vanish [11]. For example, for greyscale images described by advected functions $I(x, t)$, the momentum is constrained to be normal to the image: $\delta l / \delta u \cdot \nabla l = 0$. Hence, the Noether quantities define the geometry of the shape space.

One of the “holy grails” in the field of variational numerical methods is to find Eulerian discretisations of fluid dynamics that arise from a variational principle. Amongst other things, this would provide the possibility of discrete forms of the Noether's Theorem described in this paper. One direction that we have previously explored is to try to find a discretisation of the diffeomorphism group and to obtain some form of reduction by symmetry; this approach has been developed in some detail, making extensive use of discrete exterior calculus, for the case of incompressible flows in [29]. In [7], it was shown that the spatial discretisation of the Lie bracket must satisfy the closure property if a reduction is to be obtained; it was also shown how space-time discretisations could be obtained in this case. Another direction that we have explored is using Clebsch constraints to enforce the evolution of the back-to-labels map [8]. This leads to a multisymplectic formulation of fluid dynamics that can be discretised by a standard recipe but reduction (elimination of the back-to-labels map) is not possible after discretisation due to symmetry breaking.

On the Hamiltonian side, the conserved quantities associated via Noether's theorem with relabelling symmetry comprise the Casimir functions. The variational derivatives of the Casimir functions are null eigenvectors of the Lie-Poisson Hamiltonian structure that arises from the Euler-Poincaré framework upon Legendre transforming. This is explained further in [16, 12, 1]. See also [27, 28] for related discussions and additional references. The conservation laws that follow from Noether's theorem for relabelling symmetry of the Eulerian fluid variables generate steady flows when substituted into the augmented Lie-Poisson brackets that include the particle labels as functions of time and spatial coordinate [12, 1].

Thus, Lie-Poisson brackets with the Casimir functions leave the Eulerian fluid variables invariant, but they shift the fluid particle labels along steady flows. This fact led to a strategy for proving nonlinear stability of equilibrium flows that was first recognized in [2] for ideal incompressible planar flows and was later developed and applied more widely in plasma physics in [16]. Likewise, the Eulerian vector fields for relabelling symmetries found here could just as well have been obtained by solving for the null eigenvectors of the Lie-Poisson bracket on the Hamiltonian side. Critical points of the sum of Hamiltonian and the Casimirs are steady equilibrium flows. The stability of these equilibrium flows may be studied by taking a second variation and determining the conditions on the equilibrium that would make the corresponding linearly conserved second variation sign definite [16].

Remark 20 (Noether's other theorem). *As mentioned earlier, Noether's original paper actually contains two theorems. The second one is generally regarded as the more subtle of the two. Noether's second theorem leads in principle to dependence among the Euler-Lagrange equations (Bianchi identities). However, for ideal fluids, we have not found any strictly Eulerian conservation laws in addition those already discussed here by using Noether's second theorem in the Euler-Poincaré context.*

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References

- [1] H. D. I. Abarbanel and D. D. Holm. Nonlinear stability of inviscid flows in three dimensions: Incompressible fluids and barotropic fluids. *Phys. Fluids*, 30:3369–3382, 1987.
- [2] V. I. Arnold. Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Sov. Math.*, 6:773–777, 1965.
- [3] V. I. Arnold and B. A. Khesin. *Topological methods in hydrodynamics*. Springer-Verlag, 1998.
- [4] D. Bak, D. Cangemi, and R. Jackiw. Energy-momentum conservation in gravity theories. *Phys. Rev. D*, 49(10):5173–5181, 1994.
- [5] T. B. Benjamin and P. J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125:137–185, 1982.
- [6] A. J. Brizard. Noether derivation of exact conservation laws for dissipationless reduced-uid models. *Phys. Plasmas*, 17:112503, 2010.
- [7] C. J. Cotter and D. D. Holm. Continuous and discrete Clebsch variational principles. *Found. Comput. Math.*, 9(2):221–242, 2009.
- [8] C. J. Cotter, D. D. Holm, and P. E. Hydon. Multisymplectic formulation of fluid dynamics using the inverse map. *Proc. Roy. Soc. A*, 463, 2007.
- [9] R. L. Dewar. Hamilton's principle for a hydromagnetic fluid with a free boundary. *Nuclear Fusion*, 18:1541–1553, 1978.
- [10] H. Ertel. Ein neuer hydrodynamischer Wirbelsatz. *Met. Z., Braunschweig*, 59:277–281, 1942.
- [11] F. Gay-Balmaz and T. S. Ratiu. Clebsch optimal control formulation in mechanics. *J. Geom. Mech.*, 3(1):41–79, 2011.
- [12] D. D. Holm. Lyapunov stability of ideal compressible and incompressible fluid equilibria in three dimensions. In D. D. Holm, J. E. Marsden, and T. S. Ratiu, editors, *Hamiltonian Structure and Lyapunov Stability for Ideal Continuum Dynamics*, pages 125–208. Univ. Montreal Press, 1986.
- [13] D. D. Holm. Euler-Poincaré dynamics of perfect complex fluids. In Paul Newton, Philip Holmes, and Alan Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 169–180. Springer New York, 2002.
- [14] D. D. Holm and J. E. Marsden. Momentum maps and measure valued solutions of the Euler-Poincaré equations for the diffeomorphism group. *Progr. Math.*, 232:203–235, 2004. <http://arxiv.org/abs/nlin.CD/0312048>.
- [15] D. D. Holm, J. E. Marsden, and T. S. Ratiu. The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. in Math.*, 137:1–81, 1998. <http://arxiv.org/abs/chao-dyn/9801015>.
- [16] D. D. Holm, J. E. Marsden, T. S. Ratiu, and A. Weinstein. Nonlinear stability of fluid and plasma equilibria. *Physics Reports*, 123:1–116, 1985.
- [17] D. D. Holm, J. T. Ratnanather, A. Trouné, and L. Younes. Soliton dynamics in computational anatomy. *NeuroImage*, 23:170–178, 2004.
- [18] P. E. Hydon and E. L. Mansfield. Extensions of Noether's second theorem: from continuous to discrete systems. [arXiv:1103.3267v1](https://arxiv.org/abs/1103.3267v1).
- [19] Y. Kosmann-Schwarzbach. *Les Théorèmes de Noether*. Éditions de École Polytechnique, Palaiseau, France, 2004. English translation, 2011.
- [20] C. Muriel, J. L. Romero, and P. J. Olver. Variational C^∞ symmetries and Euler-Lagrange equations. *J. Diff. Eq.*, 222:164–184, 2006.
- [21] E. Noether. Invariante variations probleme. *Nachr. König. Gessell. Wissen. Göttingen, Mathphys. Kl.*, pages 235–257, 1918. See Transport Theory and Stat. Phys. 1 (1971) 186–207 for an English translation, which is also posted at <http://arxiv.org/pdf/physics/0503066>.

- [22] P. J. Olver. Conservation laws in elasticity. I. General results. *Arch. Rational Mech. Anal.*, 85:119–129, 1984.
- [23] P. J. Olver. Conservation laws in elasticity. II. Linear homogeneous isotropic elastostatics. *Arch. Rational Mech. Anal.*, 85:131–160, 1984.
- [24] P. J. Olver. Conservation laws in elasticity. III. Planar linear anisotropic elastostatics. *Arch. Rational Mech. Anal.*, 85:167–181, 1984.
- [25] P. J. Olver. Noether's theorems and systems of Cauchy-Kovalevskaya type. In B. Nicolaenko, D. D. Holm, and J. M. Hyman, editors, *Nonlinear Systems of Partial Differential Equations in Applied Mathematics*, volume 23, pages 81–104. Amer. Math. Soc, Providence, R.I., 1986.
- [26] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer, New York, 1993.
- [27] N. Padhye and P. J. Morrison. Fluid element relabeling symmetry. *Phys. Lett. A*, 219:287–292, 1996.
- [28] N. Padhye and P. J. Morrison. Relabeling symmetries in hydrodynamics and magnetohydrodynamics. *Plasma Phys. Rep.*, 22:869–877, 1996.
- [29] D. Pavlov, P. Mullen, Y. Tong, E. Kanso, J. E. Marsden, and M. Desbrun. Structure-preserving discretization of incompressible fluids. *Physica D*, 240(6):333–458, 2011.
- [30] P. L. Similon. Conservation laws for relativistic guiding-center plasma. *Phys. Lett. A*, 112(1):33–37, 1985.
- [31] D. E. Soper. *Classical Field Theory*. Wiley, New York, 1976.